AN EXTREMAL PROBLEM ON POTENTIALLY $K_{r+1} - (kP_2 \cup tK_2)$-GRAPHIC SEQUENCES

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ABSTRACT. A graphic sequence S is potentially $K_m - H$-graphical if it has a realization containing a $K_m - H$ as a subgraph. Let $\sigma(K_m - H, n)$ denote the smallest degree sum such that every n-term graphical sequence S with $\sigma(S) \geq \sigma(K_m - H, n)$ is potentially $K_m - H$-graphical. In this paper, we determine $\sigma(K_{r+1} - (kP_2 \bigcup tK_2), n)$ for $n \geq 4r + 10, r + 1 \geq 3k + 2t, k + t \geq 2, k \geq 1, t \geq 0$. To now, the problem of determined $\sigma(K_{r+1} - H, n)$ for H not containing a cycle on 3 vertices and sufficiently large n has been solved.

1. Introduction

The set of all non-increasing nonnegative integer sequences $\pi = (d_1, d_2, ..., d_n)$ is denoted by NS_n. A sequence $\pi \epsilon NS_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. A graphical sequence π is potentially H-graphical if there is a realization of π containing H as a subgraph, while π is forcibly H-graphical if every realization of π contains H as a subgraph. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1}-graphic. Let $\sigma(\pi) = d(v_1) + d(v_2) + ... + d(v_n)$, and $[x]$ denote the largest integer less than or equal to x. If G and G_1 are graphs, then $G \cup G_1$ is the disjoint union of G and G_1. If $G = G_1$, we abbreviate $G \cup G_1$ as $2G$. We denote $G + H$ as the graph with $V(G+H) = V(G) \bigcup V(H)$ and $E(G+H) = E(G) \bigcup E(H) \bigcup \{xy : x \in V(G), y \in V(H)\}$. Let K_k, C_k, T_k, and P_k denote a complete graph on k vertices, a cycle on k vertices, a tree on $k+1$ vertices, and a path on $k+1$ vertices, respectively. Let $K_m - H$ be the graph obtained from K_m by removing the edge set $E(H)$ of the graph H (H is a subgraph of K_m).

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdös [2] in 1938 and in general by Turán [20]. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is

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forcibly H-graphical. Here we consider the following variation: determine the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is potentially H-graphical. We denote this minimum l by $\sigma(H,n)$. Erdös, Jacobson and Lehel [4] showed that $\sigma(K_k,n) \geq (k-2)(2n-k+1)+2$ and conjectured that equality holds. They proved that if π does not contain zero terms, this conjecture is true for $k=3,\ n\geq 6$. The conjecture is confirmed in [5],[15],[16],[17] and [18]. Recently, Ferrara, Gould and Schmitt [7] proved the conjecture using graph theoretic techniques. Ferrara, Gould and Schmitt [8] determined in $\sigma(F_k,n)$ where F_k denotes the graph of k triangles intersecting at exactly one common vertex.

Gould, Jacobson and Lehel [5] also proved that $\sigma(pK_2,n)=(p-1)(2n-2)+2$ for $p\geq 2$; $\sigma(C_4,n)=2[\frac{3n-1}{2}]$ for $n\geq 4$. Luo [19] characterized the potentially C_k graphic sequences for $k=3,4,5$. Gupta, Joshi and Tripathi [6] gave a necessary and sufficient condition for the existence of a tree of order n with a given degree set. Meng-Xiao Yin, Jian-Hua Yin [28] characterize the potentially (K_5-e)-positive graphic sequences and give two simple necessary and sufficient conditions for a positive graphic sequence π to be potentially K_5-graphic, where K_r is a complete graph on r vertices and K_r-e is a graph obtained from K_r by deleting one edge. Moreover, they also give a simple necessary and sufficient condition for a positive graphic sequence π to be potentially K_6-graphic. Gould et al. [5] determined $\sigma(K_{2,2},n)$ for $n\geq 4$. Yin et al. [23-26] determined $\sigma(K_{r,s},n)$ for $s\geq r\geq 2$ and sufficiently large n. Lai [10] determined $\sigma(K_4-e,n)$ for $n\geq 4$. Yin,Li and Mao[22] determined $\sigma(K_{r+1}-e,n)$ for $r\geq 3,\ r+1\leq n\leq 2r$ and $\sigma(K_5-e,n)$ for $n\geq 5$. Yin and Li[21] gave a good method (Yin-Li method) of determining the values $\sigma(K_{r+1}-e,n)$ for $r\geq 2$ and $n\geq 3r^2-r-1$ (In fact, Yin and Li[21] also determining the values $\sigma(K_{r+1}-ke,n)$ for $r\geq 2$ and $n\geq 3r^2-r-1$). After reading[21], using Yin-Li method Yin [27] determined $\sigma(K_{r+1}-K_3,n)$ for $n\geq 3r+5, r\geq 3$. Lai [11] determined $\sigma(K_5-K_3,n)$ for $n\geq 5$. Lai [12,13] determined $\sigma(K_5-C_4,n),\sigma(K_5-P_3,n)$ and $\sigma(K_5-P_4,n)$, for $n\geq 5$. Determining $\sigma(K_{r+1}-H,n)$, where H is a tree on 4 vertices is more useful than a cycle on 4 vertices (for example, $C_4\not\subset C_i$, but $P_3\subset C_i$ for $i\geq 5$). So, after reading[21] and [27], using Yin-Li method Lai and Hu[14] determined $\sigma(K_{r+1}-H,n)$ for $n\geq 4r+10, r\geq 3, r+1\geq k\geq 4$ and H be a graph on k vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices and $\sigma(K_{r+1}-P_2,n)$ for $n\geq 4r+8, r\geq 3$. In this paper, using Yin-Li method we prove the following two theorems.

Theorem 1.1. If $r\geq 4$ and $n\geq 4r+10$, then $\sigma(K_{r+1}-(P_2\bigcup K_2),n)=(r-1)(2n-r)-2(n-r)$.

Theorem 1.2. If $n\geq 4r+10, r+1\geq 3k+2t, k+t\geq 2, k\geq 1, t\geq 0$, then $\sigma(K_{r+1}-(kP_2\bigcup tK_2),n)=(r-1)(2n-r)-2(n-r)$.

To now, the problem of determined $\sigma(K_{r+1}-H,n)$ for H not containing a cycle on 3 vertices and sufficiently large n has been solved.

2. Preparations

In order to prove our main result,we need the following notations and results.

Let $\pi=(d_1,\cdots,d_n)\epsilon NS_n, 1\le k\le n$. Let

$$\pi_k''=\begin{cases}(d_1-1,\cdots,d_{k-1}-1,d_{k+1}-1,\cdots,d_{d_k+1}-1,d_{d_k+2},\cdots,d_n),\\ \text{ if } d_k\ge k,\\ (d_1-1,\cdots,d_{d_k}-1,d_{d_k+1},\cdots,d_{k-1},d_{k+1},\cdots,d_n),\\ \text{if } d_k<k.\end{cases}$$

Denote $\pi_k'=(d_1',d_2',\cdots,d_{n-1}')$,where $d_1'\ge d_2'\ge\cdots\ge d_{n-1}'$ is a rearrangement of the $n-1$ terms of π_k''. Then π_k' is called the residual sequence obtained by laying off d_k from π.

Theorem 2.1[21] Let $n\ge r+1$ and $\pi=(d_1,d_2,\cdots,d_n)\epsilon GS_n$ with $d_{r+1}\ge r$. If $d_i\ge 2r-i$ for $i=1,2,\cdots,r-1$, then π is potentially A_{r+1}-graphic.

Theorem 2.2[21] Let $n\ge 2r+2$ and $\pi=(d_1,d_2,\cdots,d_n)\epsilon GS_n$ with $d_{r+1}\ge r$. If $d_{2r+2}\ge r-1$, then π is potentially A_{r+1}-graphic.

Theorem 2.3[21] Let $n\ge r+1$ and $\pi=(d_1,d_2,\cdots,d_n)\epsilon GS_n$ with $d_{r+1}\ge r-1$. If $d_i\ge 2r-i$ for $i=1,2,\cdots,r-1$, then π is potentially $K_{r+1}-e$-graphic.

Theorem 2.4[21] Let $n\ge 2r+2$ and $\pi=(d_1,d_2,\cdots,d_n)\epsilon GS_n$ with $d_{r-1}\ge r$. If $d_{2r+2}\ge r-1$, then π is potentially $K_{r+1}-e$ -graphic.

Theorem 2.5[9] Let $\pi=(d_1,\cdots,d_n)\epsilon NS_n$ and $1\le k\le n$. Then $\pi\epsilon GS_n$ if and only if $\pi_k'\epsilon GS_{n-1}$.

Theorem 2.6[3] Let $\pi=(d_1,\cdots,d_n)\epsilon NS_n$ with even $\sigma(\pi)$. Then $\pi\epsilon GS_n$ if and only if for any t,$1\le t\le n-1$,

$$\sum_{i=1}^{t}d_i\le t(t-1)+\sum_{j=t+1}^{n}min\{t,d_j\}.$$

Theorem 2.7[5] If $\pi=(d_1,d_2,\cdots,d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Lemma 2.1 [27] If $\pi=(d_1,d_2,\cdots,d_n)\epsilon NS_n$ is potentially $K_{r+1}-e$-graphic, then there is a realization G of π containing $K_{r+1}-e$ with the $r+1$ vertices $v_1,\cdots,v_{r+1}$ such that $d_G(v_i)=d_i$ for $i=1,2,\cdots,r+1$ and $e=v_rv_{r+1}$.

Lemma 2.2 [14] Let $n\ge r+1$ and $\pi=(d_1,d_2,\cdots,d_n)\epsilon GS_n$ with $d_r\ge r-1$ and $d_{r+1}\ge r-2$. If $d_i\ge 2r-i$ for $i=1,2,\cdots,r-2$, then π is potentially $K_{r+1}-P_2$-graphic.

Lemma 2.3 [14] Let $n\ge 2r+2$ and $\pi=(d_1,d_2,\cdots,d_n)\epsilon GS_n$ with $d_{r-2}\ge r$. If $d_{2r+2}\ge r-1$, then π is potentially $K_{r+1}-P_2$-graphic.

3. Proof of Main results.

Lemma 3.1. If $n\ge r+1, r+1\ge 3k+2t, k+t\ge 2, k\ge 1, t\ge 0$, then $\sigma(K_{r+1}-(kP_2\bigcup tK_2),n)\ge(r-1)(2n-r)-2(n-r)$.

Proof. Let

$$G=K_{r-2}+\overline{K_{n-r+2}}$$

Then G is a unique realization of $((n-1)^{r-2},(r-2)^{n-r+2})$ and G clearly does not contain $K_{r+1}-(kP_2\bigcup tK_2)$, where the symbol x^y means x repeats y times in the sequence. Thus $\sigma(K_{r+1}-(kP_2\bigcup tK_2),n)\geq(r-2)(n-1)+(r-2)(n-r+2)+2=(r-1)(2n-r)-2(n-r)$.

The Proof of Theorem 1.1 According to Lemma 3.1, it is enough to verify that for $n\geq 4r+10$,

$$\sigma(K_{r+1}-(P_2\bigcup K_2),n)\leq(r-1)(2n-r)-2(n-r).$$

We now prove that if $n\geq 4r+10$ and $\pi=(d_1,d_2,\cdots,d_n)\epsilon GS_n$ with

$$\sigma(\pi)\geq(r-1)(2n-r)-2(n-r),$$

then π is potentially $K_{r+1}-(P_2\bigcup K_2)$-graphic.

If $d_{r-2}\leq r-1$.

(1)Suppose $d_{r-2}=r-1$ and $\sigma(\pi)=(r-3)(n-1)+(r-1)(n-r+3)$, then $\pi=((n-1)^{r-3},(r-1)^{n-r+3})$. Obviously π is potentially $K_{r+1}-(P_2\bigcup K_2)$ graphic.

(2)Suppose $d_{r-2}=r-1$ and $\sigma(\pi)<(r-3)(n-1)+(r-1)(n-r+3)$, then

$$\begin{array}{rcl}\sigma(\pi)&<&(r-3)(n-1)+(r-1)(n-r+3)\\&=&(r-1)(n-1)-2(n-1)+(r-1)(n-r+3)\\&=&(r-1)(2n-r)-2(n-r),\end{array}$$

which is a contradiction.

(3)Suppose $d_{r-2}<r-1$, then

$$\begin{array}{rcl}\sigma(\pi)&<&(r-3)(n-1)+(r-1)(n-r+3)\\&=&(r-1)(n-1)-2(n-1)+(r-1)(n-r+3)\\&=&(r-1)(2n-r)-2(n-r),\end{array}$$

which is a contradiction.

Thus, $d_{r-2}\geq r$ or π is potentially $K_{r+1}-(P_2\bigcup K_2)$ graphic.

If $d_r\leq r-2$, then

$$\begin{array}{rcl}\sigma(\pi)&=&\sum_{i=1}^{r-1}d_i+\sum_{i=r}^{n}d_i\\&\leq&(r-1)(r-2)+\sum_{i=r}^{n}min\{r-1,d_i\}+\sum_{i=r}^{n}d_i\\&=&(r-1)(r-2)+2\sum_{i=r}^{n}d_i\\&\leq&(r-1)(r-2)+2(n-r+1)(r-2)\\&=&(r-1)(2n-r)-2(n-r)-2\\&<&(r-1)(2n-r)-2(n-r),\end{array}$$

which is a contradiction. Hence $d_r\geq r-1$.

If $d_{r+1}\leq r-3$.

(1)Suppose $d_r=n-1$, then $d_1\geq d_2\geq\cdots\geq d_{r-1}\geq d_r=n-1$, therefore $d_1=d_2=\cdots=d_r=n-1$. Therefore $d_{r+1}\geq r$, which is a contradiction.

(2)Suppose $d_r \le n-2$, then

$$\begin{array}{rcl}\sigma(\pi) & = & \sum_{i=1}^{r-1} d_i + d_r + \sum_{i=r+1}^{n} d_i \\ & \le & (r-1)(r-2) + \sum_{i=r}^{n} min\{r-1, d_i\} + d_r + \sum_{i=r+1}^{n} d_i \\ & = & (r-1)(r-2) + min\{r-1, d_r\} + d_r + 2\sum_{i=r+1}^{n} d_i \\ & \le & (r-1)(r-2) + 2d_r + 2\sum_{i=r+1}^{n} d_i \\ & \le & (r-1)(r-2) + 2(n-2) + 2(n-r)(r-3) \\ & = & (r-1)(2n-r) - 2(n-r) - 2 \\ & < & (r-1)(2n-r) - 2(n-r),\end{array}$$

which is a contradiction.

Thus $d_{r+1} \ge r-2$.

If $d_i \ge 2r-i$ for $i = 1,2,\cdots,r-2$ or $d_{2r+2} \ge r-1$, then π is potentially $K_{r+1} - (P_2 \bigcup K_2)$ graphic($\pi = ((n-1)^{r-3},(r-1)^{n-r+3})$) or π is potentially $K_{r+1} - P_2$-graphic by Lemma 2.2 or Lemma 2.3 . Therefore, π is potentially $K_{r+1} - (P_2 \bigcup K_2)$-graphic. If $d_{2r+2} \le r-2$ and there exists an integer i, $1 \le i \le r-2$ such that $d_i \le 2r-i-1$, then

$$\begin{array}{rcl}\sigma(\pi) & \le & (i-1)(n-1) + (2r+1-i+1)(2r-i-1) \\ & & +(r-2)(n+1-2r-2) \\ & = & i^2 + i(n-4r-2) - (n-1) \\ & & +(2r-1)(2r+2) + (r-2)(n-2r-1).\end{array}$$

Since $n \ge 4r+10$, it is easy to see that $i^2 + i(n-4r-2)$, consider as a function of i, attain its maximum value when $i = r-2$. Therefore,

$$\begin{array}{rcl}\sigma(\pi) & \le & (r-2)^2 + (n-4r-2)(r-2) - (n-1) \\ & & +(2r-1)(2r+2) + (r-2)(n-2r-1) \\ & = & (r-1)(2n-r) - 2(n-r) - n + 4r + 9 \\ & < & \sigma(\pi),\end{array}$$

which is a contradiction.

Thus, $\sigma(K_{r+1} - (P_2 \bigcup K_2), n) \le (r-1)(2n-r) - 2(n-r)$ for $n \ge 4r+10$.

The Proof of Theorem 1.2 By Lemma 3.1, for $n \ge 4r+10, r+1 \ge 3k+2t, k+t \ge 2, k \ge 1, t \ge 0$, $\sigma(K_{r+1} - (kP_2 \bigcup tK_2), n) \ge (r-1)(2n-r) - 2(n-r)$. Obviously, for $n \ge 4r+10, r+1 \ge 3k+2t, k+t \ge 2, k \ge 1, t \ge 0$, $\sigma(K_{r+1} - (kP_2 \bigcup tK_2), n) \le \sigma(K_{r+1} - (P_2 \bigcup K_2), n)$. By theorem 1.1, for $n \ge 4r+10, r \ge 4$, $\sigma(K_{r+1} - (P_2 \bigcup K_2), n) = (r-1)(2n-r) - 2(n-r)$. Then $\sigma(K_{r+1} - (kP_2 \bigcup tK_2), n) = (r-1)(2n-r) - 2(n-r)$, for $n \ge 4r+10, r+1 \ge 3k+2t, k+t \ge 2, k \ge 1, t \ge 0$.

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